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Chaotic advection and heat transfer enhancement in Stokes flows

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Abstract

The heat transfer rate from a solid boundary to a highly viscous fluid can be enhanced significantly by a phenomenon which is called *chaotic advection or Lagrangian turbulence*. Although the flow is laminar and dominated by viscous forces, some fluid particle trajectories are chaotic due either to a suitable boundary displacement protocol or to a change in geometry. As in turbulent flow, the heat transfer rate enhancement between the boundary and the fluid is intimately linked to the mixing of fluid in the system. Chaotic advection in *real* Stokes flows, i.e. flows governed by viscous forces and that can be constructed experimentally, is reviewed in this paper. An emphasis is made on recent new results on 3-D time-periodic open flows which are particularly important in industry.

Keywords: Chaotic advection; Mixing; Viscous fluids; 3-D periodic flow

1. Introduction

Chaotic advection is a subject which has attracted some attention over the past two decades. It can be described simply as chaotic particle trajectories in a laminar flow dominated by viscous forces. This phenomenon can occur in two-dimensional time-periodic or in three-dimensional flows. The equations of motion can be written as a two or a three equation autonomous dynamical system which exhibits chaotic behaviour. The first known study on such a dynamical system is due to Henri Poincaré (1893) and his work on the three body problem. From classical mechanics it is known that the orbits of planets around the sun are elliptical, as discovered by Kepler in the early seventeenth century from an analysis of astronomical data gathered over a very long period of time. Using Newton's laws, elliptical orbits of planets around the sun can also be demonstrated mathematically when only the attractive force between the sun and the planet is considered. Now, what happens when the planet is subject to other forces, such as the attraction of other planets? In the solar system, only the gravitational force between Jupiter and Saturn (because of their enormous mass) can be considered as non-negligible compared to the attractive forces between planets and the sun. It has been shown that the orbits of Jupiter and Saturn are indeed chaotic, although one needs to monitor the orbits for a very long time, of the order of several thousand of years, to notice such behaviour.

Recent interest in chaotic dynamical systems was sparked by Lorenz (1963) and his now well known and famous three equation model of convective movement in the atmosphere. Having written a computer program to solve his three equation dynamical system (a very simplified model of the earth's climate) from initial data, Lorenz discovered that his obtained results were totally different when the input data was double precision (with 16 significant figures) from those obtained with single precision input data. In fact, chaotic behaviour is often defined as *sensitivity to initial conditions*. This explains why the catch phrase *The flap of a butterfly wing in Brazil can cause a tornado in Europe* has become so popular. Notice that in the two chaotic dynamical systems discussed so far the time scale necessary to observe such behaviour is very different. In the first case, one needs to look at data over several thousands of years to notice chaotic behaviour while, in the second case, a week is often long enough.

The particular topic of chaotic advection in flows governed by viscous forces starts with the publication of Aref's paper in 1984. There, the author explains that the advection problem in a Lagrangian representation

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defines a dynamical system. If the fluid is incompressible and the flow is two-dimensional the system is Hamiltonian and has just one degree of freedom. Mixing in a stirred tank is studied using an idealized model. The agitators are modeled as point vortices and the fluid is assumed inviscid. Chaotic trajectories arise when the agitator is moved in such a way that the potential flow is unsteady.

As mentioned above, it can be shown that chaotic advection in Stokes flows is possible only if the flow is 2-D with periodic boundary conditions or three-dimensional. There are several *real* flow configurations which give rise to such behaviour. The most simple two-dimensional flow configuration is the one between two eccentric rotating cylinders. When either or both cylinders are turning at *constant* angular velocities, it is possible to solve the Stokes equations and obtain a closed form analytical solution. This is the starting point of all mathematical analysis on this Hamiltonian system. Obviously, the particle trajectories are not chaotic when the cylinders turn at constant angular velocity. However, if one cylinder is made to turn at an angular velocity which varies with time, then the equations become non-integrable and chaotic particle trajectories are possible. For the unsteady case, a closed form analytical solution is no longer available. Nevertheless, if the process is assumed to be quasi-static, i.e. if the modulation is slow, then one can assume that the flow field is given by the instantaneous angular velocity ratio. The dynamical system thus obtained can be integrated numerically using any well known numerical technique. When the cylinders turn in opposite directions, the instantaneous streamlines show that a homoclinic saddle point appears in the region of minimum gap. The disruption of saddle connections is a well-established mechanism for the generation of chaos in Hamiltonian systems (Rom-Kedar et al., 1990; Ottino, 1990).

There are other two-dimensional real periodic flow configurations which exhibit chaotic particle trajectories. The flow between two confocal ellipses whose walls glide circumferencially was imagined (by analogy to the motion of planets around the sun, see Ekeland (1984)) and constructed in France. For the case where the walls are moving in opposite directions, the streamlines exhibit a heteroclinic tra-jectory with two saddle points connected by two different streamlines. Although much more difficult to construct, it can be shown that this flow geometry can give better mixing/heat transfer enhancement than the flow between eccentric rotating cylinders simply because there are twice as many saddle points. Other 2-D periodic flow configurations which have been studied in the literature are cavity flows, and the vortex mixing flow (two cylinders inside a larger one, all three can turn).

The three-dimensional flow configurations where chaotic advection has been studied can be separated into two categories, steady and time periodic flows. Some authors prefer the terminology of spatially periodic and time periodic 3-D flows. The 3-D steady flow which exhibits chaotic advection is the one in a twisted pipe. This flow geometry consists of a series of segments whose plane of curvature forms an angle relative to the preceding segment. This flow can mix fluid over its cross section without any moving parts. And the pressure drop does not significantly increase by this phenomenon. However, in this flow with no moving parts, there is no control parameter and mixing performance depends solely on the total number of bends in the tube.

If an axial flow is superposed into any 2-D periodic flow configuration discussed above, chaotic advection can also occur. This type of flow is sometimes called a $2\frac{1}{2}$-D flow because the three velocity components are independent of the axial coordinate. In these systems the degree of mixing and the heat transfer enhancement can be controlled somewhat by the modulation frequency of the boundary displacement. This advantage is gained at the expense of moving parts.

In this review paper we examine each flow category separately. Obviously 2-D periodic flows are considered first since most results, theoretical and experimental, deal with these flows. The other two categories, 3-D steady flows and 3-D periodic flows are considered later. For each flow category, once chaotic behaviour has been observed, we highlight the analytical, numerical and experimental tools that have been developed recently to quantify chaos and mixing. In a certain respect, until the end of the 1980s, most engineering research was concerned with showing that the studied system did exhibit chaotic behaviour and with the development of tools for the study of chaotic dynamical systems. Beginning in the 1990s, the developed tools for the quantification of chaos have been applied to the real flows in order to optimize mixing and heat transfer enhancement.

2. Heat transfer enhancement in 2-D flows

To understand physically the phenomenon of chaotic heat transfer enhancement, let us first consider fluid placed between two concentric circular cylinders. When the boundaries are made to turn at very low angular velocity so that inertial effects are not important, the solution of the equations of motion leads to an angular velocity profile that is a function of radius. This flow is one-dimensional. If a blob of tracer is put anywhere in the fluid and if one or both cylinders are made to turn at very low angular velocity so that again inertial effects are negligible, one may see that, although the tracer appears to mix inside the fluid, this is not the case. If the direction of the velocities is inverted so that the cylinders are returned to their initial position one can see, as in the film by G.I. Taylor *Low Reynolds numbers hydrodynamics*, that the tracer comes back to its original position and only the effects of molecular diffusion alter the initial picture. In this very well-known film, this experiment was set up to demonstrate that Stokes flows are *reversible*. Time-periodic operation (as long as inertial effects are negligible) changes nothing to the picture.

From a heat transfer point of view, the same lackluster results are observed. Let us suppose that the inner and outer cylinders are maintained at temperatures T_{hot} and T_{cold} respectively. If the two cylinders are motionless, the steady-state temperature distribution will be exactly given by the solution of $\nabla^2 T = 0$, or in cylindrical coordinates by $T(r) = C_1 \ln r + C_2$. Now, if one or both cylinders are made to turn slowly, the temperature distribution *does not change* so long as creeping flow conditions are maintained since the advection terms $\mathbf{V} \cdot \nabla T = 0$ are equal to zero. When the two cylinders are concentric, for the heat transfer rate to increase one must turn the cylinders fast enough to create Taylor vortices, a subject not studied here. Notice that the wall to fluid local heat flux which is simply $\mathbf{q} = -\lambda \nabla T$ remains the same as in the motionless case and the Nusselt number, which is defined as the convective to the conductive heat flux, is exactly equal to 1. In conclusion, the effort spent in turning the cylinders is, in this case, absolutely worthless since the heat transfer rate (or the mixing of the tracer) is not increased by the cylinders' motion.

2.1. Journal bearing flow

Now, what happens if the two same cylinders are placed in an *eccentric* position. Notice that this geometry is very similar to the previous one since only the inner cylinder location has changed. In this case *the flow field is two-dimensional*. Because of its importance in lubrication, this problem has attracted much attention over the last 120 years, and even Osborne Reynolds (1886) himself derived a critical eccentricity necessary for separation to occur when the inner cylinder only is in motion. The analytical solution of the Stokes equations in this geometry has been obtained in at least three different coordinate systems. The natural choice here is a *bipolar, orthogonal coordinate system*. In this coordinate system, Jeffery (1922), Müller (1942) and Ballal and Rivlin (1976) have all obtained a closed form analytical solution. The mathematical expressions of all the constants of integration (they are very long) are only given in the last paper above. Unfortunately, bipolar coordinates do not degenerate into cylindrical coordinates as the eccentricity tends towards zero. Furthermore, there is also a singularity problem as the eccentricity tends towards 1. For this reason, Di Prima and Stuart (1972) obtained the solution to this same problem using a *modified bipolar coordinate system* which degenerates into cylindrical coordinates as the eccentricity tends to zero. The mathematical expressions are, nevertheless very long and tedious. To be concise, a third analytical solution with a mixed Cartesian-polar non-orthogonal coordinate system, was obtained by Wannier (1950).

Although analytical solutions for this problem have appeared in the literature in different coordinate systems, they all give the same results. Here the streamlines are more complicated than in the concentric case because the flow field is two-dimensional. Fig. 1, from Ottino (1989), shows typical streamlines for the 4 possible angular velocity displacements, (a) inner cylinder motion, (b) outer cylinder motion, (c) counter-rotation and (d) co-rotation. In each case, a vortex appears in the streamline pattern. At first thought, one can be led to assume that these 2-D steady flows mix well. Actually, the flow field is such that the flow domain is separated into two or more zones. If a tracer is put into a given zone it remains in that zone. Notice also the appearance of saddle points (points where two streamlines intersect) when the cylinders are both turning, cases (c) and (d). This is an important feature (Peixoto, 1962) since time-periodic operation leads to the disruption of homoclinic or heteroclinic trajectories. This is important because this phenomenon is responsible for enabling the tracer to go from one zone to the other, i.e. chaotic advection.

In the journal bearing flow, the heat transfer rate from the wall to the fluid when the two boundaries turn at constant velocity can be calculated by solving the convection/diffusion equation numerically. Unlike the concentric case, the advection terms $\mathbf{V} \cdot \nabla T$ terms are not zero here. Assuming that the hydrodynamic and thermal problems are not coupled (for example via temperature dependent fluid properties or natural

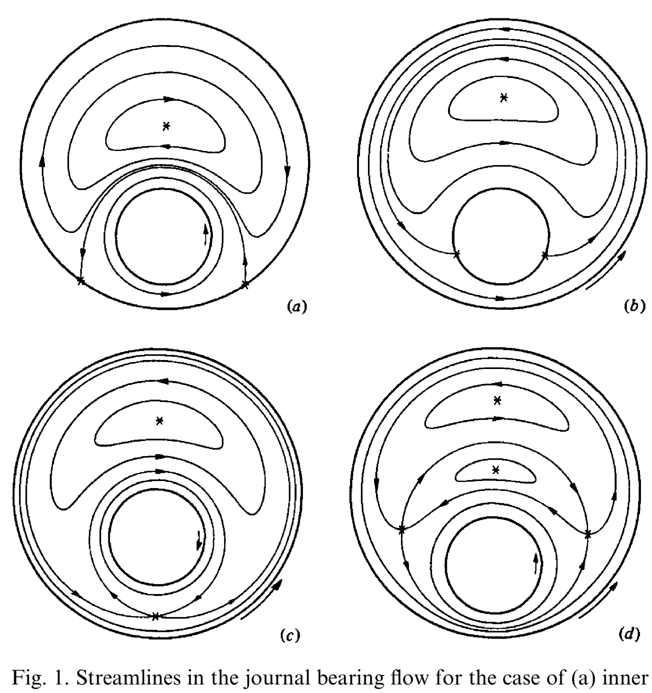

Fig. 1. Streamlines in the journal bearing flow for the case of (a) inner cylinder rotation, (b) outer cylinder rotation, (c) counter-rotation, (d) co-rotation.

convection), it is possible to use the analytical velocity field to solve for the thermal field in the conservation of energy equation. Defining, as before, a Nusselt number as the heat transfer rate divided by the heat transfer rate when both cylinders are motionless (pure conduction), numerical simulations show that the obtained Nusselt number depends on the square root of the Péclet number, $Nu \propto Pe^{1/2}$. Here, since the boundaries turn at constant velocity, the heat transfer enhancement is due to the recirculation region, it depends on the size of the vortex; the bigger the vortex the higher the average Nusselt number.

Once the two-dimensional flow field for constant angular velocities is known, what happens when tracer is put in the annular region and the cylinders are turned, not at constant angular velocity but with an angular velocity which varies periodically with time. In order for the tracer to mix, at least one cylinder must be made to turn at an angular velocity which varies time-periodically while the other one turns at a constant angular velocity. Another possibility is to turn one cylinder at a time. This experiment was constructed during the 1980s by Chaiken et al. (1986) in Columbia University in order to show, visually, chaotic advection in Stokes flows.

For any boundary displacement protocol, the trajectories of all material points are either periodic or chaotic. This is illustrated in Fig. 2, reprinted from the article by Swanson and Ottino (1990). It shows a journal bearing system and two blobs of coloured dye, one of red dye and the other of blue dye. The blobs of dye are placed initially (a) in the region of maximum gap. The cylinders are turned alternatively one at a time. One period consists of an inner cylinder rotation of 450° and an outer cylinder rotation of 150° in the opposite direction. The photographs (b)–(i) were taken after 1, 2, 3, 4, 8, 12, 16 and 30 periods. Notice how the red dye

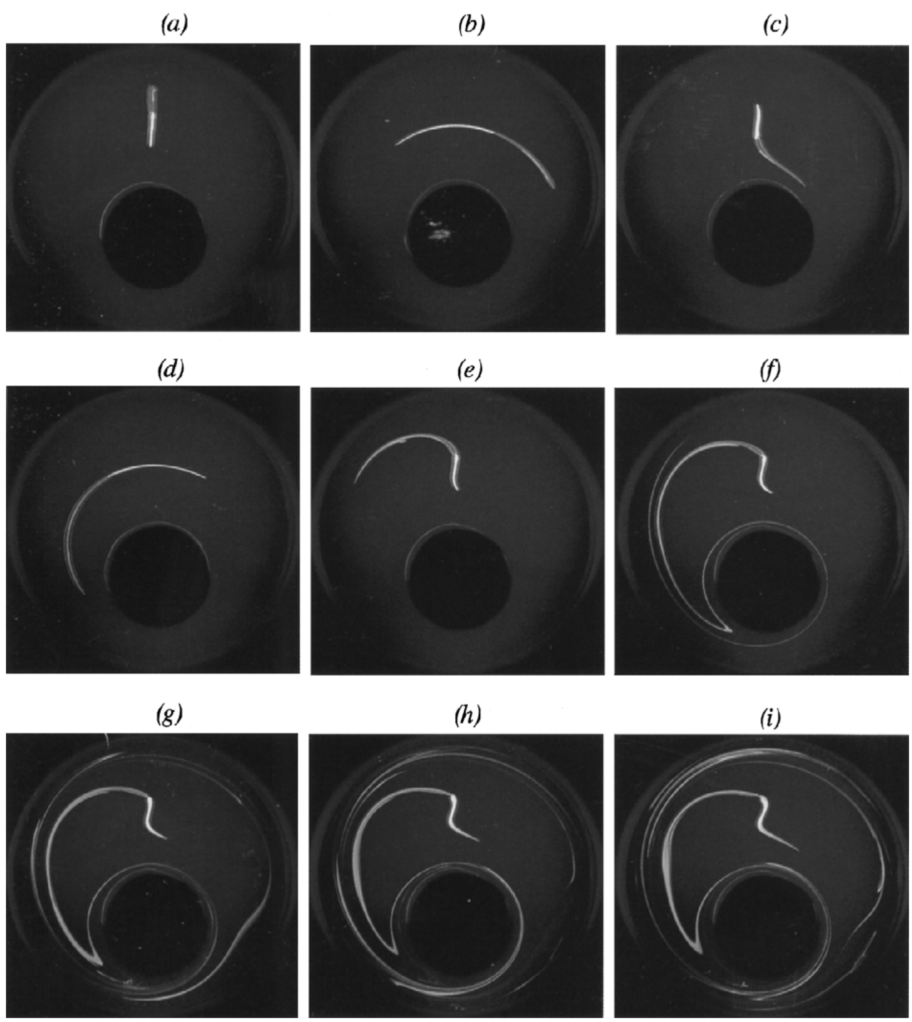

Fig. 2. Experiments demonstrating the near solid-body rotation in the regular region and the large difference in stretching between the regular and the chaotic regions. The geometry is defined by $R_{out}/R_{in} = 3$ and $\epsilon = 0.45$. The outer cylinder displacement per period is $\theta+ = 150°$ and the inner cylinder displacement per period is three times as much. The initial placement of the two line segments (a) and the deformation after 1, 2, 3, 4, 8, 12, 16, and 20 periods (b)–(i) respectively. This figure appeared in Swanson and Ottino (1990).

stretches and folds, its length increases exponentially. On the other hand, the white dye does not mix even after such a large number of periods because it was placed initially in a regular region.

Now, is it possible to predict whether fluid placed at a certain location will mix thoroughly for a given boundary movement without doing the experiments? The studies by Chaiken et al. (1987) and by Aref and Balachandar (1986), which appeared at around the same time, calculated Poincaré sections, one of the most simple tools available for the analysis of 2-D periodic flows, for this flow. As shown clearly in Aref (1984), the Lagrangian representation of 2-D unsteady incompressible flow can be written as a two equation, non-autonomous dynamical system of the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x, y, z, t), \quad i = 1, 2$$

which can be integrated numerically. A material point is chosen anywhere in the flow domain and its position at the end of 1, 2, 3, 4,... periods is plotted on a plane. If the positions spread chaotically, then the initial condition is chaotic and the tracer put there will stretch and fold and mix thoroughly. On the other hand, if a material point was initially placed in a periodic point, the tracer will move to different locations and, after a number of periods, will return to its initial position. This occurs because the tracer was placed initially in a regular region. If placed in a *regular* region the tracer will not stretch and fold and thus will not mix well. In industrial applications, one is interested in rate information, i.e. how fast does the tracer mix in the fluid domain. Poincaré sections are an interesting tool but they only give an indication of long term behaviour.

A major paper dealing with mixing in the journal bearing flow is the work by Swanson and Ottino (1990). The authors constructed an experimental setup (Fig. 2 is from their work) and developed several analytical and numerical tools to study chaos in this flow. Apart from Poincaré sections, they also found the locations of low order periodic points. A very good agreement with Poincaré sections was obtained because low order elliptic periodic points were centers of regular islands. These authors also plotted the unstable manifolds of the perturbed saddle point (counter-rotating case) in the region of minimum gap. The agreement between these plots and the blob deformation experiments is remarkable for different boundary displacement protocols. Another tool developed there is the calculation of the stretching field. To calculate the stretching of an infinitesimal vector of arbitrary initial orientation one must know the deformation tensor. Since, for this flow, the analytical solution for constant boundary conditions is known, the length stretch can be calculated at discrete points in the flow. An arbitrary value of stretching is chosen for which mixing is considered *good*. Plotting the well mixed region with a given colour and badly mixed regions with another colour the authors obtain an excellent agreement between stretching plots and both experiments and other analytical tools developed in the paper.

One must keep in mind that a characteristic feature of all Stokes flows is that they are *reversible*. Even if the tracer is placed in a chaotic region, one can return the system to its original position by inverting the angular displacements. This property can be used to test both the experimental setup and the numerical integration technique employed.

How is the heat transfer rate affected by time-periodic boundary movement? Again if the inner and outer boundaries are kept at T_{hot} and T_{cold} respectively, when both cylinders are motionless, the solution of $\nabla^2 T = 0$ leads to the solution, in bipolar coordinates $[\alpha, \beta, z]$:

$$T = C_1\alpha + C_2,$$

where C_1 and C_2 are integration constants. This equation gives the temperature distribution in the journal bearing for the pure conduction regime.

When one or both cylinders are in motion at constant angular velocity, the temperature of fluid inside the circulation zone becomes practically homogeneous due to the mixing process there. This leads to a heat transfer rate increase between the boundary and the fluid. One can expect to obtain a Nusselt number enhancement of about 50% depending both on the size of the vortex zone and on the Péclet number. In their work, Ghosh et al. (1992) showed that the Nusselt number is proportional to $Pe^{1/2}$ for constant angular velocities. Since the heat transfer enhancement over the conduction solution is due to the vortex, the enhancement is a maximum in the region of maximum gap. In the region where the gap is thinnest, the streamlines are nearly circular and the temperature profile is practically linear from one wall to the other, there is no heat transfer enhancement here. To increase further the heat transfer enhancement, one must find a way of exchanging fluid from the vortex zone to the other regions. This can be done by varying the angular velocity of one cylinder periodically in time.

For the journal bearing flow, Ghosh et al. (1992) considered the counter-rotating case and allowed the inner cylinder angular velocity to vary sinusoidally as a function of time. Obviously to increase the heat transfer enhancement the amplitude of the modulation must be as high as possible so that the instantaneous angular velocity ratio varies over the widest possible range as long as Stokes flow conditions are valid in the flow. The authors used the Melnikov method and an analysis by Chirikov (1979) to show that an optimum modulation frequency exists for which heat transfer is a maximum. The authors tested their theoretical results by solving numerically the convection/diffusion equation in order to calculate the heat transfer enhancement in the journal bearing flow. Their numerical code gave the same value of the optimum modulation frequency. However, the enhancement was somewhat lower than that predicted by the analysis.

The journal bearing geometry is defined by two dimensionless parameters, the clearance ratio and the eccentricity ratio. So far, in all calculations we have assumed that the journal bearing geometry was given. However one may ask if there is a particular value of the eccentricity ratio for which mixing and heat transfer enhancement are best? Kaper and Wiggins (1993) applied tools from the field of adiabatic dynamical systems theory to make quantitative predictions of quantities in quasi-steady Stokes flows with slowly varying saddle stagnation points i.e. the counter-rotating journal bearing flow. They show that a large amplitude continuous slow and periodic modulation of the angular velocity causes the flow to exhibit large non-integrable regions. They define a potential mixing zone from the maximum and minimum instantaneous stagnation streamlines and show that this zone covers the largest part of the flow domain when the cylinders are almost concentric (low eccentricity) and when the amplitude of the modulation varies over the widest range. They also obtain a value of the optimum modulation frequency which is somewhat smaller than the value predicted by Ghosh et al. (1992).

All experimental and theoretical studies discussed up to now have dealt with Newtonian fluids. In practice, the highly viscous fluids for which chaotic advection can be an interesting alternative if they are to be heated or cooled in a shorter amount of time are usually non-Newtonian. Niederkorn and Ottino (1993) constructed an apparatus very similar to the one built by Swanson and performed dye advection experiments with viscoelastic fluids. In order to calculate the different analytical tools, they also obtained a numerical solution to the steady flow using a finite difference method. Their algorithm used a special form of the compressible continuity equation to form a purely hyperbolic set of equations. As in Swanson and Ottino (1990), the comparison between experimental and numerical results of blob advection are impressive. The effects due to the elasticity of the fluid are not obvious. In some cases the size of regular islands increases with the Weissenberg number while in other cases the opposite effect occurs. A more recent study by Kumar and Homsy (1996) is an attempt at explaining this not well understood phenomenon.

3. Other 2-D periodic flow geometries

Chien et al. (1986) had previously studied chaotic advection and mixing experimentally in cavity flows. In this flow configuration, a rectangular enclosure containing fluid has both its top and bottom walls which can move. By an alternate periodic motion of the two walls, these authors show how a blob of fluid stretches and folds exponentially. After several periods the blob can reach a length of several meters. The influence of boundary displacement protocol and of initial location are also discussed in this paper. A novel mapping method developed by Kruijt et al. (2001) uses this flow as an example to investigate the influence of geometry and operating conditions on mixing quality.

A more complicated flow occurs when two cylinders are placed inside a larger one, all three can rotate. This apparatus was constructed by Jana et al. (1994). For constant angular velocities of all cylinders, there are many more possible streamline portraits in this flow. The authors also obtained numerically the solution for steady boundary conditions using a boundary integral method technique in order to develop dynamical tools for use in mixing studies. They compare experiments with numerical calculations and use a variety of tools such as the calculation of stretching distribution, the location of periodic points, and the calculation of the unstable manifolds to analyze mixing in this flow. Here the authors show that higher order periodic points can be even more important than period-one points in establishing the advection template and extended regions of large stretching. Another major point is that as long as the forcing function produces the same displacement per period, it produces the same qualitative mixing pattern.

Another two-dimensional flow which has been constructed in order to study mixing is the one between two confocal ellipses. In this flow, the inner and outer boundaries glide along their circumference, the flow geometry does not change with time. This geometry possesses two axis of symmetry while the journal bearing flow possesses only one. Symmetries are very important in the study of mixing, as pointed out by Franjione et al. (1989). Large islands are located on lines of symmetry or on opposite sides of the line. It is possible to manipulate symmetries in such a way so that an island is moved into a region of good mixing.

The first step consisted in finding the analytical solution of the creeping flow equations in elliptical coordinates (Saatdjian et al. (1994)). When the two ellipses are gliding in opposite directions, a heteroclinic trajectory (two saddle points connected by two different streamlines) is formed in the flow. By varying the instantaneous angular velocity ratio, the saddle points are displaced from one wall to the other. At the same time, the experimental setup was constructed, (see Saatdjian et al. (1995, 1996)) and dye advection experiments were successfully compared to numerical simulations. Heat transfer calculations (Leprévost and Saatdjian, 1998) using the analytical velocity distribution to obtain the temperature field showed that better mixing was obtained when both boundaries were turning at the same time than when they were made to turn alternatively.

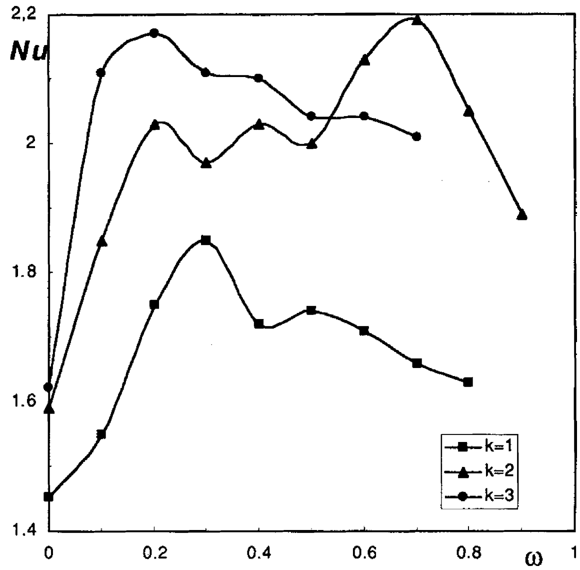

Fig. 3. Heat transfer enhancement as a function of modulation frequency in the 2-D periodic flow between confocal elliptic cylinders.

None of the tools discussed so far could have predicted such a result. Fig. 3 shows the heat transfer enhancement in this geometry as a function of the modulation frequency. An appropriate choice of the modulation frequency can lead to a heat transfer enhancement over the pure conduction regime of more than 200%. Although it is difficult to compare the performance of two different geometries precisely, it is clear that this geometry yields better results than the journal bearing flow because it contains twice as many saddle points which upon disruption enable fluid to move from one zone to the other.

4. Heat transfer enhancement in 3-D steady flows

There are not very many examples of *real* 3-D flows dominated by viscous forces and where chaotic particle trajectories can be observed. Steady three-dimensional flows have been known to give rise to chaotic streamlines ever since the work of Hénon (1966) on the ABC flow, the three equation dynamical system is:

$$\frac{dx}{dt} = A\sin z + C\cos y, \quad \frac{dy}{dt} = B\sin x + A\cos z,$$
$$\frac{dz}{dt} = C\sin y + B\cos x.$$

Hénon showed that this flow contained KAM tori as well as chaotic motions of the Smale horseshoe type. However, the above flow is a steady solution of the 3-D Euler equations, the flow itself has little practical importance. In the chemical industry, mixing of highly viscous fluids is sometimes obtained by placing internals in a tube. One of the most popular is the Kenics static mixer, it is widely used in industry. Khakhar et al. (1987) imagined a *partitioned pipe mixer*, as a model of this mixer, they show that it is possible to obtain chaotic trajectories in this steady 3-D flow. The mixer consists of a pipe partitioned into a sequence of semi-circular ducts by means of rectangular plates fixed perpendicularly to each other. Fluid is forced through the tube by an axial pressure gradient while the pipe is rotated about its axis relative to the assembly of plates. A cross sectional flow is thus created in each semi-circular element. Using an approximate velocity field obtained using the method of weighted residuals, the authors plotted Poincaré sections and calculated local stretching of material lines. The authors conclude with a series of questions, remained unanswered even today, both on the practical side and on the fundamental nature of the flow. This mixer was constructed later and visualization experiments were performed by Kusch and Ottino (1992).

The theoretical work on another real, steady 3-D flow exhibiting chaotic behaviour by Jones et al. (1987, 1990) was probably the catalyst that was needed to spark several research teams in Europe and in the United States to undertake research on this area and to construct the flow imagined by these authors. This flow is simpler to construct than the partitioned pipe mixer because it contains no moving parts. A twisted pipe with fluid flowing inside it was imagined. In order to avoid confusion, the term curved denotes a section of pipe that has a constant radius of curvature and lies in a plane. Curved pipes can be imagined as parts of a torus. On the other hand, *twisted* pipes consist of curved pipe segments which are not all in the same plane. It is well known that in a curved pipe, inertia leads to the formation of two longitudinal vortices of opposite sign. The considered pipe geometry was composed of a series of curved segments. The twist arises because the plane of curvature of each subsequent pipe segment forms an angle relative to the preceding segment. One can define a pitch angle χ which is the twist angle between two curved segments. A pitch angle $\chi = 0$ represents a torus and $\chi = 180$ represents an S-shaped pipe. Both of these cases are not very interesting because the twisted pipe is confined to a plane. For a given pitch angle between these two limits, the symmetry planes of the secondary vortices in successive bends will not coincide and a particle flowing down the pipe will experience a sequence of transverse flows. This process creates chaotic particle motion.

To show that chaotic streamlines occur in such a twisted pipe, Jones et al. (1987, 1990) derive an approximate velocity field based on the earlier analysis of Dean (1927, 1928) for a curved pipe. The flow of particles in a twisted pipe of pitch angle χ is represented as a sequence of Dean solutions augmented by a rotation of the particles through an angle $-\chi$ between successive elements. Some very restrictive assumptions are implied in the analysis. The flow is assumed to be fully developed when it enters the curved segment, the flow is assumed to readjust very quickly from one secondary flow to the other. And finally, the lowest-order Dean solution is employed even though higher order approximations are available.

Nevertheless, the above authors show, using different tools such as Poincaré sections, that chaotic streamlines do occur in this flow. The practical implication is that the stirring quality is enhanced without requiring additional energy input into the system. The wall to fluid heat transfer coefficient is also increased by the twisted pipe geometry without a notable increase in pressure drop. The authors conclude that although their model is not very realistic for a twisted pipe consisting of a succession of basic cells, they argue that it can be approximated in the laboratory by inserting segments of straight pipe between curved elements. The authors openly invite researchers to address this flow from an experimental point of view.

In the beginning of the last decade, flow visualization experiments (Peerhossaini and Le Guer, 1990; Le Guer and Peerhossaini, 1991) in a curved pipe using a laser induced fluorescence technique have confirmed the existence of chaotic trajectories in this steady flow. Peerhossaini et al. (1993) constructed a Plexiglas curved channel with 4 bends for these experiments, see Fig. 4. The pipe has a square cross-section and a twist angle of $\xi = \pi/2$. Each curved segment is separated from the next one by a straight element. Axial velocity measurements at the entrance and exit of each bend for Dean numbers comprised between 141 and 530 were performed. For the lowest Dean number the results show that even though the flow is fully developed at the entrance of the first curved element, the Dean vortices are not completely developed at the exit. Dye advection experiments on this installation clearly show the characteristics of this 3-D chaotic flow. Dye was injected at the entrance via a square grid containing 81 holes. For some injection locations, the dye stretches and folds exponentially, at the end of 4 bends the length of the filament is several orders of magnitude greater than at the entrance. For other injection locations the stretching is regular. This behaviour is very similar to the one encountered in 2-D periodic flows discussed above.

In conjunction with the visualization experiments, these authors constructed a heat exchanger test facility. Two experimental coils, one helical and the other *chaotic* were constructed from stainless steel tubing. Both coils had the same length and the same number of bends. The coiled tubes were plunged into a constant temperature bath. Entrance and exit temperatures were measured for each case. Obviously, great care was taken in order to test the two coils for exactly the same conditions. Depending on the Reynolds number, the heat transfer enhancement can reach 20% despite a pressure drop increase which is very small, of the order of 1%.

These results are in total agreement with those obtained in the United States by Acharya et al. (1992). These authors also constructed a heat exchanger facility and tested two different coils, a constant axis and an alternating (chaotic) axis coil. These authors also develop a numerical model based on the low order Dean solution. These authors obtain a heat transfer

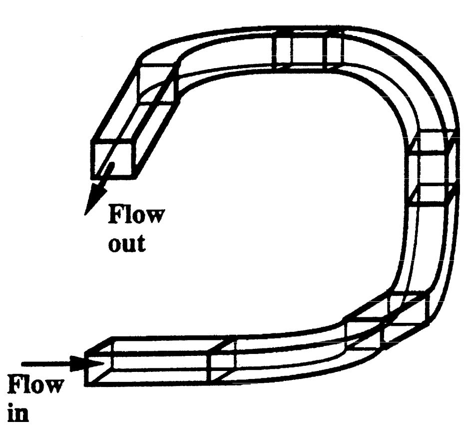

Fig. 4. Geometry used in experiments of chaotic advection in the twisted pipe flow, see Peerhossaini et al. (1993).

Fig. 5. Streaklines in the flow between confocal elliptic cylinders, the axial velocity is 1 mm/s.

enhancement and a pressure drop increase very similar to the results obtained by Peerhossaini et al. (1993). However, in both studies, the authors conclude that the flow geometry has not been optimized and that greater gains can be expected in the future.

5. Time-periodic 3-D flows

If an axial flow is super-imposed onto the 2-D time-periodic journal bearing flow, chaotic streaklines can be observed in the flow. In fact, any tangential 2-D periodic flow such as the flow between confocal ellipses with circumferencial gliding walls or the vortex mixing flow has a 3-D periodic counterpart. Fig. 5 shows the streakline obtained between confocal elliptic cylinders, the outer cylinders turns at constant velocity while the inner cylinder turns at a time-periodic angular velocity. The fluid viscosity here is very high, the average axial velocity is about 1 mm/s and the axial Reynolds number is of the order of 0.1.

All of these flows belong to a certain particular class since the three velocity components are all independent of the axial coordinate. Concerning the flow in an eccentric helical annular mixer, the Eulerian velocity field is known analytically for steady boundary conditions. The axial velocity component is determined by solving the axial component of $\nabla P = \mu\nabla^2\mathbf{V}$ separately since this equation is not coupled to the cross-sectional components.

Kusch (1991) and Kusch and Ottino (1992) constructed the eccentric helical annular mixer, and published very beautiful photographs of chaotic streaklines. The experiments are both labor intensive and very expensive since the gallons of dyed glycerine that flow through the mixer cannot be recycled. Since the

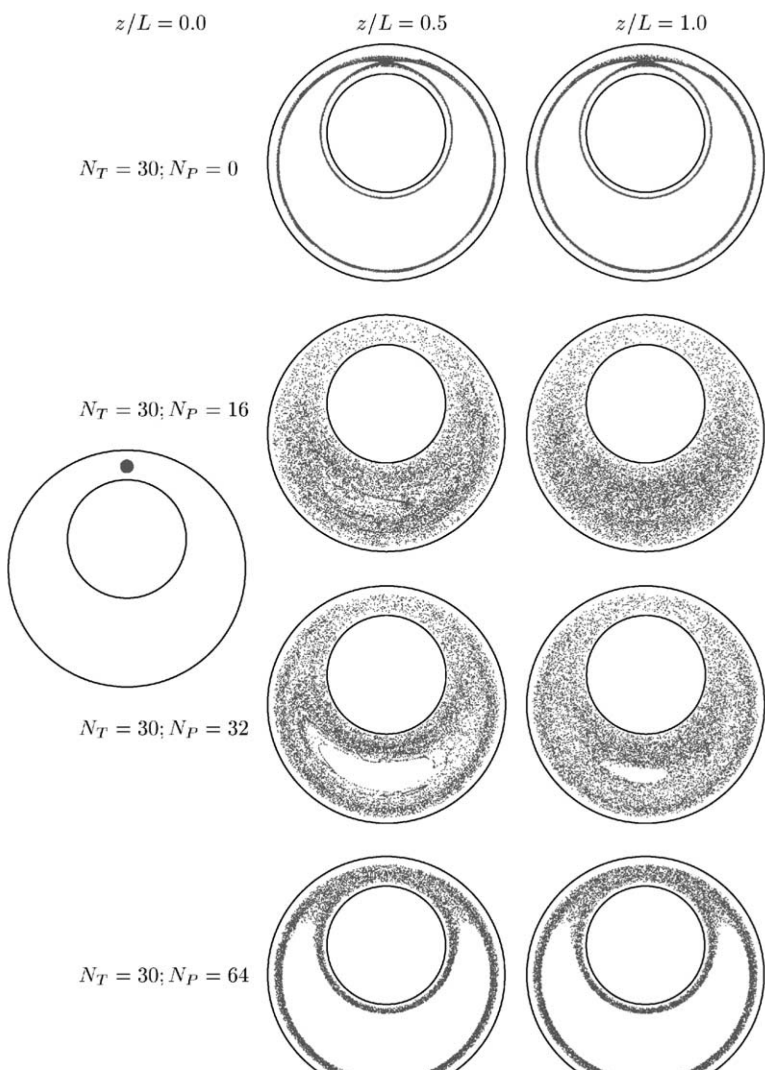

Fig. 6. Exit temperature standard deviation as a function of the number of periods of inner cylinder rotation N_P. In all cases the parameter $N_T = 16$ is fixed.

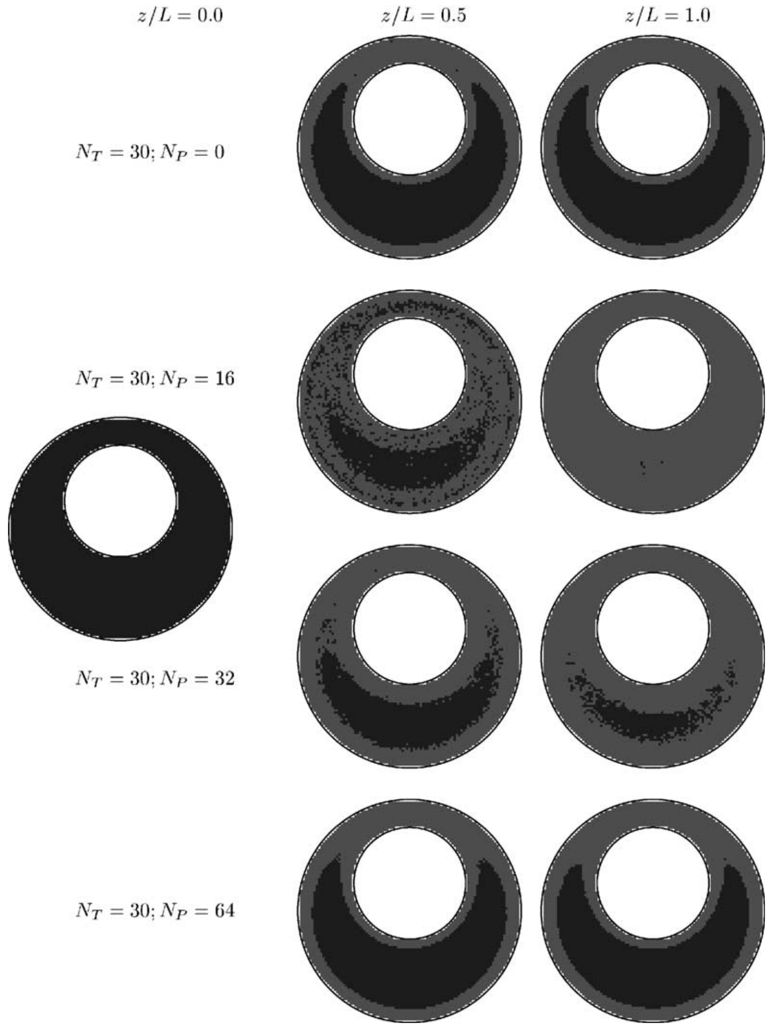

Fig. 7. Dye advection experiment in the eccentric helical annular mixer. The initial blob (figure on the left) is represented by 100,000 material points. Their exit location half way down the mixer and at the exit are plotted for different values of N_P and N_T. The case $N_T = 30$ and $N_P = 16$ gives the best heat transfer enhancement.

analytical solution involves no approximations (apart from creeping flow conditions), they were able to compare experimental and numerically calculated streaklines. This is not possible either in the partitioned pipe mixer or in the twisted duct flow because the model equations are approximate. How-ever, apart from this comparison between experiment and model equations, there is a lack of analytical tools available for the optimization of the boundary displacement protocol.

Mixing experiments in the annular region between confocal ellipses like that shown in Fig. 4 have been performed by Leprévost (2000) and Lefèvre et al. (in press). The experiments show that a blob of tracer will stay inside KAM tubes all along the reactor. The vertical tubes are defined roughly by the potential mixing zone defined by Kaper and Wiggins (1993). However, as in the experiments by Kusch (1991), the influence of the axial velocity component is seen to be very important though not well understood.

A recent study by Rodrigo et al. (in press) takes a quantitative look at the flow in the eccentric helical annular mixer. The case where the cylinders are turning in opposite directions is considered. The outer cylinder turns at a constant angular velocity while the inner cylinder turns at an angular velocity which can vary sinusoidally with time. Two dimensionless parameters are defined, N_T and N_P. The first one, N_T is a measure of the average number of turns the outer cylinder makes during the average residence time of a tracer particle in the reactor. Notice that this parameter does not depend on the modulation frequency. The second one, N_P is a measure of the number of periods the inner cylinder makes during the average residence time of a tracer particle in the reactor. Fluid enters the adiabatic mixer (or heat exchanger) with a linear, radial temperature profile. The exit temperature profile is analyzed, and in particular the standard deviation at the exit is calculated for different values of the parameters N_P and N_T.

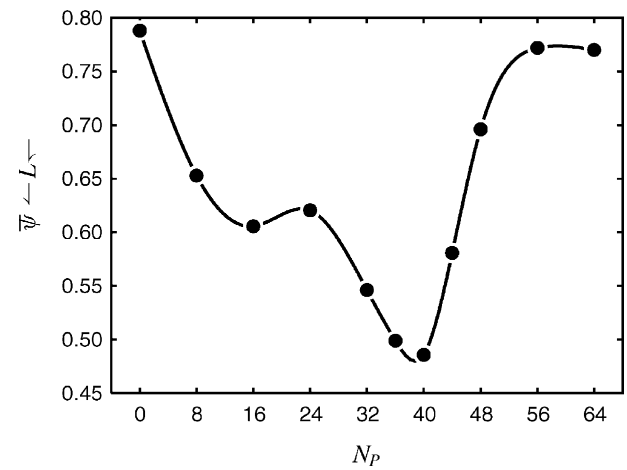

Fig. 8. Stretching distribution in the eccentric helical annular mixer. This quantity is calculated from the rate of deformation tensor. The stretching is assumed good, and coloured red, if it exceeds a given cutoff value. Notice that for the case $N_T = 30$ and $N_P = 16$, which gives the best heat transfer enhancement, the stretching is good over practically the entire cross-section.

Obviously the best boundary displacement protocol is the one which gives a uniform cross-sectional profile at the exit. Calculations clearly show that for a given value of N_T, there is a value of N_P which leads to a minimum in the exit temperature standard deviation, this is shown in Fig. 6. Other tools discussed previously confirm this result. For example, a blob of tracer is introduced at the entrance. Since the trajectories can be calculated from the analytical solution for constant boundary conditions, one can plot the exit tracer location for different values of N_P and N_T.

The case leading to the best mixing is obviously the one for which the blob spreads chaotically over the whole cross-section, as shown in Fig. 7. The calculation of the stretching field for this flow, shown in Fig. 8, also leads to the same conclusion. The mixing protocol of this apparatus can now be optimized.

6. Conclusions

The heat transfer rate into or out of highly viscous, high Prandtl number fluids can be enhanced significantly by a phenomenon known as chaotic advection or Lagrangian turbulence. In this review article we have focused on the two-dimensional periodic and three-dimensional real flows that can be used in an industrial process. The former are useful in batch processes while the latter are suited for continuous flow operation. Despite the fact that many real flows exhibiting chaotic behaviour have been imagined and constructed over the last two decades, chaotic flows have, unfortunately, not been used in industry.

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